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Chaotic Properties of Mappings on a Probability Space

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Sensitive dependence on initial conditions is widely understood as being the central idea of chaos. We first give sufficient conditions (both topological and ergodic) on an endomorphism to ensure the sensitivity property. Then, a strong sensitivity concept is introduced. Sufficient conditions on a transformation implying strong sensitivity are given. We also provide bounds for the strong sensitivity constant.

Key Words: chaos; sensitive dependence; initial conditions; endomorphism; exact; mixing.

1. INTRODUCTION

1.1. *Chaos and Sensitive Dependence on Initial Conditions*

Strong interest has recently been shown in the areas of nonlinear dynamical systems and chaos (see, e.g., Abraham et al. [1], Berliner [3], Chatterjee and Yilmaz [6], Collet [7], Glasner and Weiss [9], and Lasota and Mackey [10]). Strangely enough, there has been no universally accepted mathematical definition of chaos. As a matter of fact, chaos is associated with complex and unpredictable behavior of phenomena over time. In a

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popular text, Devaney [8] puts forward several components as being the essential features of chaos. Let $T: X \to X$ be a map on some metric space (X, d). Then

- T is *topologically transitive* if for every pair of non-empty open sets U and V in X, there is an integer $n \geq 0$ such that $U \cap T^n V \neq \varnothing$.

- T has *sensitive dependence on initial conditions* if there exists $\delta > 0$ (a *sensitivity constant*) such that for every point $x \in X$ and every open neighborhood V_x of x, there exists an integer $n \geq 0$ such that $\sup_{y \in V_x} d(T^n x, T^n y) > \delta$.

In the definition of Devaney [8], a continuous map is called *chaotic* if it is topologically transitive and sensitive and if its periodic points are dense in X. Whereas the density of periodic points can be interpreted as an "element of regularity" (Devaney [8]), the sensitivity property captures the idea that in a chaotic system a very small change in the initial condition can cause a big change in the trajectory. Recently, the definition of Devaney was simplified by Banks et al. [2]. Indeed, these authors showed that any continuous and topologically transitive map $T: X \to X$ whose periodic points are dense in X has sensitive dependence on initial conditions.

Sensitive dependence on initial conditions is widely understood as being the central idea of chaos and was popularized by the meteorologist Ed Lorenz through the so-called butterfly effect. Figure 1 illustrates the sensitivity property for the celebrated quadratic map $T: [0, 1] \to [0, 1]$ defined by $Tx = 4x(1 - x)$ (see Devaney [8], Lasota and Mackey [10], and Wegman [14]), the archetype of chaotic maps (actually, it is chaotic in the sense of Devaney). However, even if sensitivity is a central idea in chaos theory, it cannot be considered alone as a definition of chaos. Indeed, though the map $Tx = 2x$ defined on $[0, +\infty[$ is obviously sensitive, it cannot reasonably be considered as a chaotic map.

The aim of the present study is to derive general conditions (both topological and ergodic) on a transformation (also called *dynamical system* in this context) to force the sensitivity property. The paper is organized as follows. In the next subsection, we have compiled without proofs some basic definitions and results that are essential to our study. In Section 2, we give sufficient conditions for an endomorphism to be sensitive. Section 3 is devoted to the study of a stronger property than the sensitive dependence on initial conditions, namely the *strong sensitivity*.

1.2. *Definitions and Notations*

Throughout the paper, (X, d) is a metric space that we assume not to be reduced to a single point, $\mathscr{B}(E)$ denotes the Borel σ-field of any topological

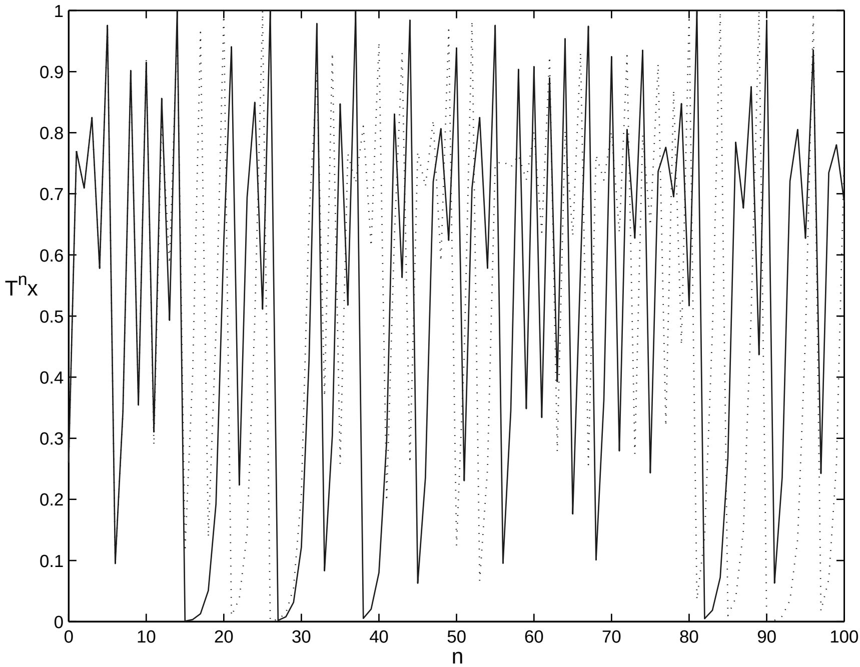

FIG. 1. First 100 iterates of the points $x = 0.26$ (solid line) and $x = 0.26001$ (dotted line) under the quadratic map $Tx = 4x(1 - x)$. Note the separation of the two trajectories after only a few iterations.

space E, and μ is a probability measure on $\big(X, \mathscr{B}(X)\big)$. We denote by $\operatorname{supp}\mu$ the support of μ.

If x is a point in X and r is a positive real number, we write $B(x, r)$ $\big($resp. $\overline{B}(x, r)\big)$ for the open ball (resp. the closed ball) with center at x and radius r. For any subset A of X, the notation $\operatorname{diam} A$ stands for the diameter of A $\big($i.e., $\sup\{d(x, y) : x \in A, y \in A\}\big)$, and the notation A^c stands for the complement of A in X. If $T\colon X \to X$ denotes a transformation on X, T^n $(n \geq 0)$ denotes the composition with itself n times (with the standard convention that T^0 is the identity map on X). A point x in X is said to be a *periodic point* (of T) if it satisfies $T^n x = x$ for some positive integer n. We say that x has *period* n if n is the smallest positive integer with this property (i.e., $T^k x \neq x$ for $k = 1, \ldots, n - 1$).

The mapping T is called *topologically mixing* (not to be confused with mixing, defined below) if for every pair of non-empty open sets U and V in X, there is an integer $N \geq 0$ such that $U \cap T^n V \neq \varnothing$ for all $n \geq N$ (Block and Coppel [5, p. 156]). Note the distinction with the definition of topological transitivity already given in the previous subsection.

Recall that a measurable mapping $T: X \to X$ is said to be an *endomorphism* on the probability space $(X, \mathcal{B}(X), \mu)$ if T is *measure-preserving*, i.e., for any $B \in \mathcal{B}(X)$, $\mu(B) = \mu(T^{-1}B)$ (see Rohlin [13]). The following four concepts describe the levels of irregularity that an endomorphism T on $(X, \mathcal{B}(X), \mu)$ can display.

- T is called *ergodic* if for any sets $A, B \in \mathcal{B}(X)$ we have
$$\frac{1}{n}\sum_{k=0}^{n-1} \mu(A \cap T^{-k}B) \to \mu(A)\mu(B) \qquad \text{as } n \to \infty.$$
- T is called *weakly mixing* if for any sets $A, B \in \mathcal{B}(X)$ we have
$$\frac{1}{n}\sum_{k=0}^{n-1} |\mu(A \cap T^{-k}B) - \mu(A)\mu(B)| \to 0 \qquad \text{as } n \to \infty.$$
- T is called *strongly mixing* if for any sets $A, B \in \mathcal{B}(X)$ we have
$$\lim_{n\to\infty} \mu(A \cap T^{-n}B) = \mu(A)\mu(B).$$
- T is called *exact* if $T\mathcal{B}(X) \subset \mathcal{B}(X)$ and if
$$\lim_{n\to\infty} \mu(T^{n}A) = 1 \qquad \text{for any } A \in \mathcal{B}(X) \text{ with } \mu(A) > 0.$$

It is a helpful and important distinction to note that exact maps, unlike each of the other types of maps described, cannot be homeomorphisms. All of these basic notions arise in ergodic theory. It can be shown that

$$\text{exact} \;\Rightarrow\; \text{strongly mixing} \;\Rightarrow\; \text{weakly mixing} \;\Rightarrow\; \text{ergodic},$$

none of these implications being an equivalence. For proofs, examples, and counterexamples, we refer the reader to Billingsley [4], Lasota and Mackey [10], Petersen [11], or Pollicott and Yuri [12].

2. SENSITIVE ENDOMORPHISMS

In this section, we consider a given endomorphism T on the metric probability space $(X, d, \mathcal{B}(X), \mu)$. Recall (see Pollicott and Yuri [12, p. 101]) that a set of integers $\mathcal{N} \subset \mathbb{N}$ is said to have *positive upper density* if

$$\limsup_{N\to\infty} \frac{1}{N+1}\, \#\{0 \le n \le N : n \in \mathcal{N}\} > 0.$$

Throughout the section, we will denote by **(P)** the following regularity property:

(P) For every non-empty open set U in X, there is an increasing sequence of nonnegative integers $(\varphi(k))_{k\ge 0}$ with positive upper density such that

$$U \cap \bigcap_{k \geq 0} T^{-\varphi(k)} U \neq \emptyset.$$

Property (**P**) means that for any non-empty open set U in X, one can always find a point in U whose trajectory infinitely often returns to U with an assumption on the return frequency. Theorem 2.1 below is the main result of this section.

THEOREM 2.1. *Assume that* $\operatorname{supp} \mu = X$. *The endomorphism T has sensitive dependence on initial conditions if one of the three statements holds*:

(i) *T is topologically mixing*;

(ii) *T is weakly mixing and property* (**P**) *is satisfied*;

(iii) *T^n is ergodic for all $n \geq 1$, and the periodic points of T are dense in X.*

We start with a technical lemma.

LEMMA 2.1. *Assume that the endomorphism T does not have sensitive dependence on initial conditions and that* $\operatorname{supp} \mu = X$. *Then there exist two non-empty disjoint open sets U and V in X and an increasing sequence of non-negative integers $(\varphi(k))_{k \geq 0}$ such that for all $k \geq 0$, $U \cap T^{\varphi(k)} V = \emptyset$. If we assume further that assumption* (**P**) *is satisfied* (*resp. that the periodic points of T are dense in X*), *then the sequence $(\varphi(k))_{k \geq 0}$ can be chosen with positive upper density* (*resp. periodic*).

The proof of Lemma 2.1 is based on the following two ideas. First, if the transformation is not sensitive, there exists a non-empty open set, called V_x in the proof, such that $\operatorname{diam} T^n V_x$ remains "small" for all integers n. Second, using Poincaré and Halmos recurrence theorems, one exhibits a subset of V_x, whose iterates infinitely often return into a fixed closed ball.

Proof of Lemma 2.1. As X is not reduced to a single point, one can find $\delta > 0$ such that $\forall y \in X$, $\overline{B}^c(y, 4\delta) \neq \emptyset$. If T has not sensitive dependence on initial conditions, there exist a point $x \in X$ and an open neighborhood V_x of x such that $\forall n \geq 0$, $\forall y \in V_x$, $d(T^n x, T^n y) \leq \delta$. Thus $\forall n \geq 0$,

$$\operatorname{diam} T^n V_x \leq 2\delta. \tag{2.1}$$

As $\operatorname{supp} \mu = X$, we obviously have $\mu(V_x) > 0$ so that, according to Poincaré and Halmos recurrence theorems for endomorphisms (see Petersen [11, pp. 34 and 39]) there is an increasing sequence of nonnegative integers $(\varphi(k))_{k \geq 0}$ such that

$$V_x \cap \bigcap_{k \geq 0} T^{-\varphi(k)} V_x \neq \emptyset. \tag{2.2}$$

Let $z \in V_x \cap \bigcap_{k \geq 0} T^{-\varphi(k)} V_x$. Since z belongs to the open set V_x, one can find $\epsilon > 0$ such that $B(z, \epsilon) \subset V_x$. Now, let $u \in B(z, \epsilon)$. For all $k \geq 0$, $d(T^{\varphi(k)}u, z) \leq d(T^{\varphi(k)}u, T^{\varphi(k)}z) + d(T^{\varphi(k)}z, z) \leq 4\delta$, by inequality (2.1). Consequently,

$$T^{\varphi(k)} B(z, \epsilon) \subset \overline{B}(z, 4\delta), \qquad \forall\, k \geq 0.$$

Consider the disjoint open sets U and V defined by $U = \overline{B}^{\,c}(z, 4\delta)$ and $V = B(z, \epsilon)$. The initial choice of δ shows that U is non-empty, and, from the inclusion above, we have $\forall\, k \geq 0$, $U \cap T^{\varphi(k)} V = \emptyset$. This completes the proof of the first assertion.

Assume that property (**P**) is satisfied. We can conduct the same proof as before (the existence of the sequence $(\varphi(k))_{k \geq 0}$ in (2.2) is now deduced from (**P**)). Obviously, such a sequence has positive upper density. To achieve the proof of the lemma, observe that if the periodic points of T are dense in X, there exists in V_x a periodic point with period $p \geq 1$, and relation (2.2) holds with $\varphi(k) = kp$, $k \geq 0$. ∎

Proof of Theorem 2.1. Sensitivity to initial conditions under (i) is a straightforward consequence of Lemma 2.1 (assume that sensitivity does not hold to obtain a contradiction).

We now proceed to show that, under (ii), T exhibits the desired sensitivity property. To this end, assume that T is weakly mixing and (**P**) holds, but that T does not have sensitive dependence on initial conditions. By Lemma 2.1, there exist $U, V \in \mathscr{B}(X)$, both with positive μ-measure, and a sequence $(\varphi(k))_{k \geq 0}$, such that $\forall\, k \geq 0$, $\mu(V \cap T^{-\varphi(k)} U) = 0$ (use the measure-preserving property of T). Consequently, $\forall\, n \geq 1$,

$$\begin{aligned}\sum_{k=0}^{n-1} |\mu(V \cap T^{-k}U) - \mu(V)\mu(U)| &\geq \sum_{\substack{k=0 \\ k \in \varphi(\mathbb{N})}}^{n-1} |\mu(V \cap T^{-k}U) - \mu(V)\mu(U)| \\ &= \#\{0 \leq k \leq n-1 : k \in \varphi(\mathbb{N})\}\mu(V)\mu(U).\end{aligned}$$

Since the sequence $(\varphi(k))_{k \geq 0}$ has positive upper density, this leads to

$$\limsup_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} |\mu(V \cap T^{-k}U) - \mu(V)\mu(U)| > 0,$$

which contradicts the weak mixing property of T.

Finally, with respect to statement (iii) of the theorem, let us assume that T^n is ergodic for all $n \geq 1$ and the periodic points of T are dense in X, but that T does not have the sensitivity property. According to Lemma 2.1, there exist $U, V \in \mathscr{B}(X)$, both with positive μ-measure, and $p \geq 1$ such that $\forall\, k \geq 0$, $\mu(V \cap T^{-pk}U) = 0$. This contradicts the ergodicity of T^p. ∎

Assuming that $\operatorname{supp}\mu = X$, examination shows that any strongly mixing endomorphism is also topologically mixing. Consequently, point (i) of Theorem 2.1 leads to the following corollary:

COROLLARY 2.1. *Assume that T is strongly mixing. If* $\operatorname{supp}\mu = X$, *then T has sensitive dependence on initial conditions.*

Remark 1. The proof of Lemma 2.1 also shows that any transformation of X into itself (not necessarily measurable and measure-preserving) which is topologically mixing *and* whose periodic points are dense exhibits sensitive dependence on initial conditions. Similarly, any transformation whose iterates are topologically transitive *and* whose periodic points are dense in X is sensitive.

Remark 2. To compare assumptions (ii) and (iii) in Theorem 2.1, observe that if T is weakly mixing then so is T^n for any $n \geq 1$ (Petersen [11, p. 72]). Consequently, with respect to the ergodic properties of T, assumption (ii) is stronger than assumption (iii). However, the analytic property of T in assumption (iii) (i.e., the density of the periodic points) is stronger than the analytic property of T in assumption (ii) (i.e., (**P**)).

Remark 3. Assuming the ergodicity of T^n for all $n \geq 1$ is not enough to ensure the sensitivity to initial conditions of T. Consider indeed the case where T is an irrational rotation on the unit circle. Precisely, let $X = \{z \in \mathbb{C} : |z| = 1\}$, $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and $Te^{2i\pi\theta} = e^{2i\pi(\theta+\alpha)}$, $\theta \in \mathbb{R}$. If we denote by λ the Lebesgue measure on X, it is well known that for all $n \geq 1$, T^n is an ergodic (but not weakly mixing) endomorphism on $(X, \mathscr{B}(X), \lambda)$ (see, e.g., Pollicott and Yuri [12, p. 114]). However, T does not have sensitive dependence on initial conditions, like any isometric transformation.

3. STRONGLY SENSITIVE TRANSFORMATIONS

It was shown in Section 2 that an exact endomorphism associated with a full support probability measure exhibits sensitive dependence on initial conditions (Corollary 2.1). As a matter of fact, a stronger property holds for such an endomorphism.

DEFINITION 3.1. Let $T: X \to X$. We call T strongly sensitive if there exists $\delta > 0$ (a strong sensitivity constant) such that for every point $x \in X$ and every open neighborhood V_x of x, there exists an integer $N \geq 0$ such that for all $n \geq N$, $\sup_{y \in V_x} d(T^n x, T^n y) > \delta$.

Observe that if δ is a strong sensitivity constant for T, then so is any positive $\delta' \leq \delta$. This leads us to consider the following quantity, denoted by $\Delta(T)$:

$$\Delta(T) = \sup\{\delta : \delta \text{ is a strong sensitivity constant for } T\}.$$

DEFINITION 3.2. Let T: $X \to X$. We call T μ-weakly exact if $T\mathcal{B}(X) \subset \mathcal{B}(X)$ and if

$$\lim_{n\to\infty} \mu(T^n U) = 1 \quad \text{for every non-empty open set } U \text{ in } X \text{ with } \mu(U) > 0.$$

On one hand, Definition 3.1 strengthens the sensitivity definition in the sense that any strongly sensitive mapping is *ipso facto* sensitive. On the other hand, Definition 3.2 weakens the notion of exactness, in the sense that the underlying transformation does not necessarily preserve the measure and that the asymptotic property has now to be checked only for open sets. The following theorem links these two new concepts.

THEOREM 3.1. *Let T: $X \to X$ be a μ-weakly exact transformation. If* $\operatorname{supp} \mu = X$, *then T has strong sensitive dependence on initial conditions.*

Proof of Theorem 3.1. Let u_0 and v_0 be two distinct points in X. Set $\delta_0 = d(u_0, v_0)/4$ and assume that the μ-weakly exact transformation T is not strongly sensitive. Then, there exist a point $x \in X$ and an open neighborhood V_x of x such that $\forall N \geq 0$, there exists $n \geq N$ with $d(T^n x, T^n y) \leq \delta_0$ for all $y \in V_x$. Consequently, there exists an increasing sequence $(n_k)_{k\geq 0}$ such that $\forall k \geq 0$,

$$\operatorname{diam} T^{n_k} V_x \leq 2\delta_0. \tag{3.1}$$

By the μ-weak exactness of T,

$$\mu(T^{n_k} V_x) \to 1 \qquad \text{as } k \to \infty. \tag{3.2}$$

Let us denote by $\mathbf{1}_A$ the characteristic function of any subset A of X, i.e., $\mathbf{1}_A(x) = 1$ (resp. 0) whenever $x \in A$ (resp. $x \notin A$). We deduce from (3.2) the existence of a subsequence $(n'_k)_{k\geq 0}$ of $(n_k)_{k\geq 0}$ such that $\mathbf{1}_{T^{n'_k} V_x} \to 1$ μ-*a.s.* Let $X_0 = [\mathbf{1}_{T^{n'_k} V_x} \to 1]$. By the above statements, $\mu(X_0) = 1$. Moreover, since $\operatorname{supp} \mu = X$, any non-empty open ball in X has positive μ-measure. In particular, one can find u and v in X_0 such that $d(u, u_0) \leq \delta_0/2$ and $d(v, v_0) \leq \delta_0/2$. Thus, by the very definition of δ_0, we obtain that $d(u, v) > 2\delta_0$. As both u and v belong to X_0, we have that, for large enough k, $\mathbf{1}_{T^{n'_k} V_x}(u) = \mathbf{1}_{T^{n'_k} V_x}(v) = 1$. Therefore, for large enough k, $\operatorname{diam} T^{n'_k} V_x \geq d(u, v) > 2\delta_0$, which contradicts inequality (3.1). ∎

The following corollary, whose proof is straightforward, establishes the announced link between exactness and strong sensitivity.

COROLLARY 3.1. *Let T be an exact endomorphism on $(X, \mathcal{B}(X), \mu)$. If* $\operatorname{supp} \mu = X$, *then T has strong sensitive dependence on initial conditions.*

The next theorem provides a lower bound for the strong sensitivity constant.

THEOREM 3.2. *Let T: $X \to X$ be a μ-weakly exact transformation. Assume that* $\operatorname{supp}\mu = X$ *and that there exists a map* s: $]0, +\infty[\to [0, +\infty]$ *such that*

$$\forall \epsilon > 0, \quad \exists \eta > 0, \quad \forall A \in \mathcal{B}(X), \quad \mu(A^c) \le \eta \Rightarrow \operatorname{diam} \mathrm{A} \ge s(\epsilon). \quad (3.3)$$

Then any positive $\delta < \sup_\epsilon s(\epsilon)/2$ is a strong sensitivity constant for T, i.e., $\Delta(T) \ge \sup_\epsilon s(\epsilon)/2$.

Proof of Theorem 3.2. First note that T is strongly sensitive in accordance with Theorem 3.1. Now, let $\epsilon > 0$ and let an associated $\eta > 0$ be given by (3.3). Let x be a point in X and let V_x be any open neighborhood of x. As T is μ-weakly exact, there exists $N \ge 0$ such that $\forall n \ge N$, $\mu(T^n V_x) \ge 1 - \eta$, i.e., $\mu((T^n V_x)^c) \le \eta$. According to (3.3), we then have $\operatorname{diam} T^n V_x \ge s(\epsilon)$ for all $n \ge N$. Two cases have to be considered.

Case 1. $\sup_\epsilon s(\epsilon) = \infty$.

For every $M > 0$, there exists $\epsilon_0 > 0$ with $s(\epsilon_0) > 2M$. Thus, for every $x \in X$ and every open neighborhood V_x of x, there exists $N \ge 0$ such that $\forall n \ge N$, $\operatorname{diam} T^n V_x > 2M$. Hence, for all $n \ge N$, there exists $y \in V_x$ such that $d(T^n x, T^n y) > M$. Consequently, any $M > 0$ is a strong sensitivity constant for T.

Case 2. $s = \sup_\epsilon s(\epsilon) < \infty$.

Let $\gamma > 0$. By assumption, there exists $\epsilon_0 > 0$ such that $s(\epsilon_0) > s - \gamma$. Therefore, for every $x \in X$ and every open neighborhood V_x of x, there exists $N \ge 0$ such that $\forall n \ge N$, $\operatorname{diam} T^n V_x > s - \gamma$. Consequently, for all $n \ge N$, there exists $y \in V_x$ such that $d(T^n x, T^n y) > (s - \gamma)/2$. We conclude that any positive $\delta < s/2$ is a strong sensitivity constant for T. ∎

COROLLARY 3.2. *Assume that X is a Borel subset of $\mathbb{R}$ endowed with the standard distance and let T: $X \to X$ be a μ-weakly exact transformation. Assume, moreover, that* $\operatorname{supp}\mu = X$ *and $\lambda \ll \mu$, where λ denotes the Lebesgue measure on X. Then*

- *If* $\lambda(X) < \infty$, *one has* $\Delta(T) \ge \lambda(X)/2$.
- *If* $\lambda(X) = \infty$, *one has* $\Delta(T) = \infty$.

Proof of Corollary 3.2. The absolute continuity of λ with respect to μ implies that for any $\epsilon > 0$, there exists $\eta > 0$ such that for any $A \in \mathcal{B}(X)$, $\mu(A^c) \le \eta \Rightarrow \lambda(A^c) \le \epsilon$. Since $\lambda(A) \le \operatorname{diam} A$, we consequently have $\operatorname{diam} A \ge \lambda(A) = \lambda(X) - \lambda(A^c) \ge \lambda(X) - \epsilon$. Applying Theorem 3.2 with $s(\epsilon) = \max(0, \lambda(X) - \epsilon)$ gives the desired conclusion. ∎

Banks et al. [2] showed that any continuous and topologically transitive map $T: X \to X$ whose periodic points are dense in X has sensitive dependence on initial conditions. In the particular case where $X = [0, 1]$, one can deduce from Theorem 3.2 the following corollary, which completes the result of Banks et al. It is assumed that the metric on $[0, 1]$ is the standard one.

COROLLARY 3.3. *Let $T: [0, 1] \to [0, 1]$ be a continuous and topologically transitive transformation. Assume that $T\mathscr{B}([0, 1]) \subset \mathscr{B}([0, 1])$ and that T has a periodic point of odd period greater than* 1. *Then T has strong sensitive dependence on initial conditions and* $\Delta(T) \geq 1/2$.

Proof of Corollary 3.3. Let λ be the Lebesgue measure on $[0, 1]$. According to Corollary 3.2, one only needs to prove that T is λ-weakly exact.

Let U be a non-empty open set in $[0, 1]$. There exists an open interval, say I, contained in U. For $p \geq 2$, denote by H_p the interval $[1/p, 1 - 1/p]$. Fix $\epsilon > 0$ and $p \geq 2$ large enough to have $\lambda(H_p) \geq 1 - \epsilon$. According to Block and Coppel [5, Proposition 44 and Theorem 46], there exists $N \geq 1$ such that $\forall n \geq N$, $H_p \subset T^n I$. Hence, if $n \geq N$, $\lambda(T^n U) \geq \lambda(T^n I) \geq 1 - \epsilon$. Thus T is λ-weakly exact. ∎

Until now, only lower bounds for $\Delta(T)$ were given. The following theorem provides an upper bound in case $X = [0, 1]^p$. For convenience, we shall assume that d is the Euclidean metric, but one can consider other metrics such as the l^p's.

THEOREM 3.3. *Assume that $X = [0, 1]^p$ $(p \geq 1)$ and d is the Euclidean metric, and let T be an ergodic endomorphism on $\left([0, 1]^p, \mathscr{B}([0, 1]^p), \mu\right)$. If T has strong sensitive dependence on initial conditions and* $\operatorname{supp} \mu = [0, 1]^p$, *then* $\Delta(T) \leq \sqrt{p}/2$.

The proof is based on the following remark. If there exists a strong sensitivity constant for T greater than $\sqrt{p}/2$, then the iterates of any starting point concentrate near the boundary of the set $[0, 1]^p$, contradicting ergodicity. We first give the following lemma, whose proof is clear:

LEMMA 3.1. *Assume that d is the Euclidean metric on $[0, 1]^p$. Fix $\delta \in]1/2, 1[$ and denote by C_δ the hypercube in $[0, 1]^p$ centered at $(1/2, \dots, 1/2)$ with edge length $2\delta - 1$. Then, for every x and y in $[0, 1]^p$ with $d(x, y) > \delta\sqrt{p}$, both x and y belong to C_δ^c.*

Proof of Theorem 3.3. Fix $\delta \in]1/2, 1[$ and assume that $\delta\sqrt{p}$ is a strong sensitivity constant for T. Then, for every $x \in [0, 1]^p$ and every open neighborhood V_x of x, there exists $N \geq 0$ such that $\forall n \geq N$, there exists $y \in V_x$ with $d(T^n x, T^n y) > \delta\sqrt{p}$. Using the notations and the result of Lemma 3.1, we obtain that for every $x \in [0,1]^p$, there exists $N \geq 0$ such that $\forall\, n \geq N$, $T^n x \in C_\delta^c$. For $N \geq 0$, let E_N be defined by

$$E_N = \bigcap_{n \geq N} T^{-n} C_\delta^c.$$

Clearly, $E_N \in \mathscr{B}([0,1]^p)$, and, by the previous statements, $\bigcup_{N\geq 0} E_N = [0,1]^p$. Consequently, there exists $N_0 \geq 0$ such that $\mu(E_{N_0}) > 0$. But,

$$E_{N_0} \subset T^{-n} C_\delta^c \qquad \text{for all } n \geq N_0,$$

so that, for $n \geq N_0 + 1$,

$$\begin{aligned}\frac{1}{n}\sum_{k=0}^{n-1} \mu(E_{N_0} \cap T^{-k} C_\delta^c) &= \frac{1}{n}\sum_{k=0}^{N_0-1} \mu(E_{N_0} \cap T^{-k} C_\delta^c) + \frac{1}{n}\sum_{k=N_0}^{n-1} \mu(E_{N_0} \cap T^{-k} C_\delta^c)\\ &= \frac{1}{n}\sum_{k=0}^{N_0-1} \mu(E_{N_0} \cap T^{-k} C_\delta^c) + \frac{n-N_0}{n}\,\mu(E_{N_0}).\end{aligned}$$

Consequently,

$$\frac{1}{n}\sum_{k=0}^{n-1} \mu(E_{N_0} \cap T^{-k} C_\delta^c) \to \mu(E_{N_0}) \qquad \text{as } n \to \infty. \tag{3.4}$$

Now, since T is ergodic,

$$\frac{1}{n}\sum_{k=0}^{n-1} \mu(E_{N_0} \cap T^{-k} C_\delta^c) \to \mu(E_{N_0})\,\mu(C_\delta^c) \qquad \text{as } n \to \infty. \tag{3.5}$$

From (3.4) and (3.5), we see that $\mu(E_{N_0}) = \mu(E_{N_0})\,\mu(C_\delta^c)$. But $\mu(E_{N_0}) > 0$, and, since $\operatorname{supp}\mu = [0,1]^p$, $\mu(C_\delta^c) < 1$ (contradiction). In conclusion, any strong sensitivity constant of T is less than $\delta\sqrt{p}$ for every $\delta \in]1/2, 1[$. ∎

Combining the results of Corollary 3.1, Corollary 3.2, and Theorem 3.3 leads to the following useful result:

COROLLARY 3.4. *Assume that $X = [0,1]$ is endowed with the standard metric and let T be an exact endomorphism on $\big([0,1], \mathscr{B}([0,1]), \mu\big)$. Assume, moreover, that $\operatorname{supp}\mu = [0,1]$ and $\lambda \ll \mu$, where λ denotes the Lebesgue measure on $[0,1]$. Then T has strong sensitive dependence on initial conditions and $\Delta(T) = 1/2$.*

EXAMPLE. Let S be any exact endomorphism on $\big([0,1], \mathscr{B}([0,1]), \lambda\big)$, where λ denotes the Lebesgue measure, and let f be *any* λ-*a.s.* positive probability density on $[0,1]$. Denote by F the distribution function associated with f and by μ the probability measure on $[0,1]$ defined by

$$\mu(A) = \int_A f\,\mathrm{d}\lambda \qquad \text{for every } A \in \mathscr{B}\big([0,1]\big).$$

By Proposition 5.6.2 and Theorem 6.5.2 in Lasota and Mackey [10], the transformation $T = F^{-1} \circ S \circ F$ is an exact endomorphism on $\big([0,1], \mathscr{B}([0,1]), \mu\big)$. In accordance with Corollary 3.4, T has strong sensitive dependence on initial conditions with $\Delta(T) = 1/2$. This construction gives interesting examples of exact endomorphisms. For instance, let S: $[0, 1] \to [0, 1]$ be the *tent map* (i.e., the piecewise linear map defined by $S(0) = 0$, $S(1/2) = 1$, and $S(1) = 0$), and let

$$f(x) = \frac{1}{\pi\sqrt{x(1-x)}}, \qquad x \in (0, 1).$$

It is easy to see that T is then the quadratic map defined by $Tx = 4x(1-x)$ for $x \in [0, 1]$, and by the preceding statements T is strongly sensitive with $\Delta(T) = 1/2$.

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REFERENCES

1. C. Abraham, G. Biau, and B. Cadre, On Lyapunov exponent and sensitivity for maps of the interval, Technical Report, University Montpellier II, 2001.
2. J. Banks, J. Brooks, G. Cairns, G. Davis, and P. Stacey, On Devaney's definition of chaos, *Amer. Math. Monthly* **99** (1992), 332–334.
3. L. M. Berliner, Statistics, probability and chaos (with discussion), *Statist. Sci.* **7** (1992), 69–122.
4. P. Billingsley, "Ergodic Theory and Information," Wiley Series in Probability and Mathematical Statistics, Wiley, New York, 1965.
5. L. S. Block and W. A. Coppel, "Dynamics in One Dimension," Springer-Verlag, Berlin, 1992.
6. S. Chatterjee and M. R. Yilmaz, Chaos, fractals and statistics (with discussion), *Statist. Sci.* **7** (1992), 49–122.
7. P. Collet, Some ergodic properties of maps of the interval, *in* "Dynamical Systems" (Temuco, 1991/1992), Travaux en Cours, Vol. 52, pp. 55–91, Hermann, Paris, 1996.
8. R. L. Devaney, "An Introduction to Chaotic Dynamical Systems," Addison-Wesley, New York, 1989.
9. E. Glasner and B. Weiss, Sensitive dependence on initial conditions, *Nonlinearity* **6** (1993), 1067–1075.
10. A. Lasota and M. C. Mackey, "Chaos, Fractals, and Noise—Stochastic Aspects of Dynamics," Springer-Verlag, New York, 1994.
11. K. Petersen, "Ergodic Theory," Cambridge Univ. Press, Cambridge, UK, 1983.
12. M. Pollicott and M. Yuri, "Dynamical Systems and Ergodic Theory," Cambridge Univ. Press, Cambridge, UK, 1998.
13. V. A. Rohlin, Exact endomorphisms of Lebesgue space, *Trans. Amer. Math. Soc.* **39** (1964), 1–36.
14. E. J. Wegman, On randomness, determinism and computability, *J. Statist. Plann. Inference* **20** (1988), 279–294.